Linear algebra: vectors, matrices, dot product
Statistical Natural Language Processing 1

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Some practical remarks
(recap)

- Course web page: https://snlp1-2024.github.io (public)
  https://github.com/snlp1-2024/snlp1/ (private)
- If you haven't done already, please fill in the questionnaire on Moodle

Today's lecture

- Some concepts from linear algebra
  - Vectors
  - Dot product
  - Matrices

This is only a high-level, informal introduction/refresher.

Linear algebra

Linear algebra is the field of mathematics that studies *vectors* and *matrices*.

- A vector is an ordered sequence of numbers

$$\boldsymbol{v} = (6, 17)$$

- A matrix is a rectangular arrangement of numbers

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

- A well-known application of linear algebra is solving a set of linear equations

$$
\begin{array}{rcrcr}
2x_1 & + & x_2 & = & 6 \\
x_1 & + & 4x_2 & = & 17
\end{array}
\qquad \Longleftrightarrow \qquad
\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}
$$

Why study linear algebra?

Consider an application counting words in a document

the	and	of	to	in	...
121	106	91	83	43	...

Why study linear algebra?

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	the	and	of	to	in	...	
(	121	106	91	83	43	...	)

Why study linear algebra?

Consider an application counting words in multiple documents

	the	and	of	to	in	...
document$_1$	121	106	91	83	43	...
document$_2$	142	136	86	91	69	...
document$_3$	107	94	41	47	33	...
...	...	...	...	...	...	

You should already be seeing vectors and matrices here.

Why study linear algebra?

- Insights from linear algebra are helpful in understanding many NLP methods
- In machine learning, we typically represent input, output, parameters as vectors or matrices (or tensors)
- It makes notation concise and manageable
- In programming, many machine learning libraries make use of vectors and matrices explicitly
- In programming, vector-matrix operations correspond to loops
- 'Vectorized' operations may run much faster on GPUs, and on modern CPUs

Vectors

- Vectors are objects with a magnitude and a direction
- We represent vectors with an ordered list of numbers $\boldsymbol{v} = (v_1, v_2, \ldots v_n)$
- The number n (the number of elements or entries of the vector) is its dimension
- We often call an n dimensional vector as n-vector
- The vector of n real numbers is said to be in $\mathbb{R}^n$ ($\boldsymbol{v} \in \mathbb{R}^n$)
- Typical notation for vectors:

$$\boldsymbol{v} = \vec{v} = (v_1, v_2, v_3) = \langle v_1, v_2, v_3 \rangle = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$
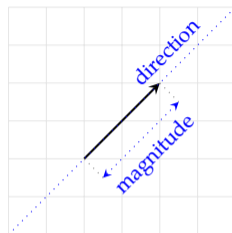
Geometric interpretation of vectors

- Geometrically, vectors are represented with arrows from the origin in the Euclidean space
- The endpoint of the vector $v = (v_1, v_2)$ correspond to the Cartesian coordinates defined by v_1, v_2
- These generally make sense for two or three-dimensional spaces
- The intuitions often (!) generalize to higher dimensional spaces
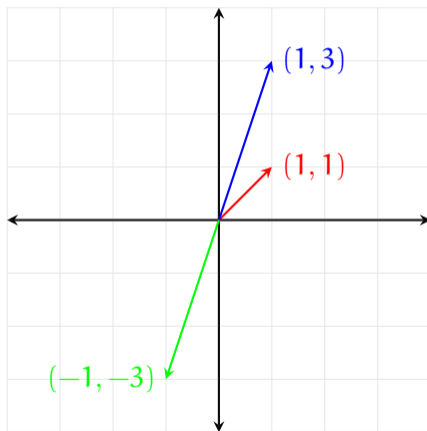
Some special vectors

- The zero vector, $\mathbf{0}$, is the vector whose all entries are 0
- The vector of all 1s, $\mathbf{1}$, is also often interesting
- A more interesting set of vectors is *standard unit vectors* (examples below are 4-dimensional standard unit vectors)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- n-dimensional standard unit vectors form the standard *basis* for n-dimensional (vector) space
- In some textbooks, standard unit vectors of two (and three) dimensions are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$
- In ML they are related to *one-hot representation*: we represent categorical predictors (variables) with n values as n-dimensional standard unit vectors

Multiplying a vector with a scalar

- For a vector $v = (v_1, v_2)$ and a scalar a,
$$av = (av_1, av_2)$$

- multiplying with a scalar 'scales' the vector

- We can use the notation $a\mathbf{1}$ for a vector whose all entries are a
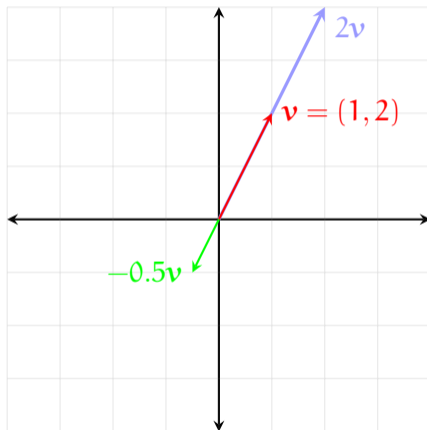
Vector addition and subtraction

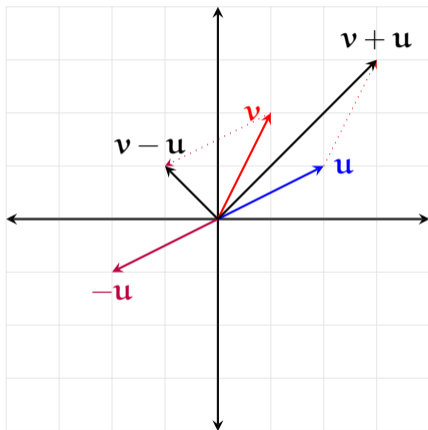For vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (w_1, w_2)$

- $\mathbf{v} + \mathbf{u} = (v_1 + w_1, v_2 + w_2)$

$$(1, 2) + (2, 1) = (3, 3)$$

- $\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u})$

$$(1, 2) - (2, 1) = (-1, 1)$$

- For any vector $\mathbf{v}$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$

Properties of vector operations

- Vector addition and scalar multiplication is commutative

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$a\mathbf{u} = \mathbf{u}a$$

- Scalar multiplication and vector addition also show the following distributive properties

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$

Linearity and linear functions

- A linear f() function (or mapping) follows
  - $f(a\mathbf{v}) = af(\mathbf{v})$ (homogeneity)
  - $f(\mathbf{v} + \mathbf{u}) = f(\mathbf{v}) + f(\mathbf{u})$ (additivity)
  - combined together: $f(a\mathbf{v} + b\mathbf{u}) = af(\mathbf{v}) + bf(\mathbf{v})$
- A combination of vectors as in

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n.$$

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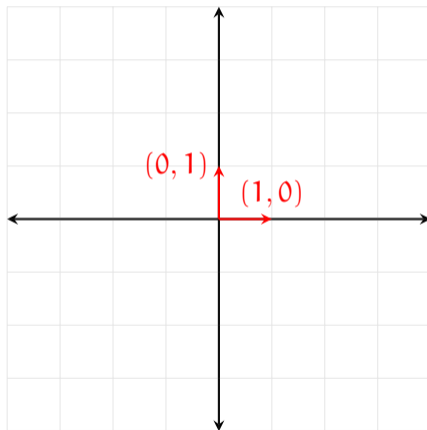

Question: Is $f(x) = ax + b$ linear?

Linear combinations of standard unit vectors

- Any n-vector can be written as a linear combination of standard unit vectors. Example:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

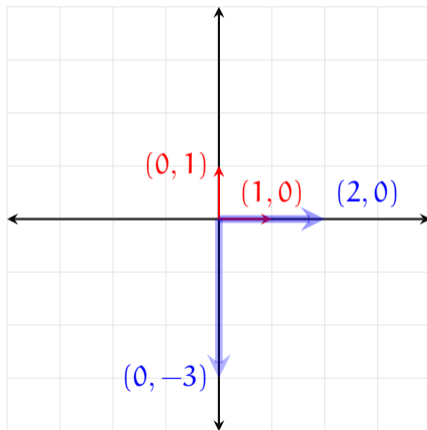- n-dimensional standard unit vectors form a *basis* for $\mathbb{R}^n$

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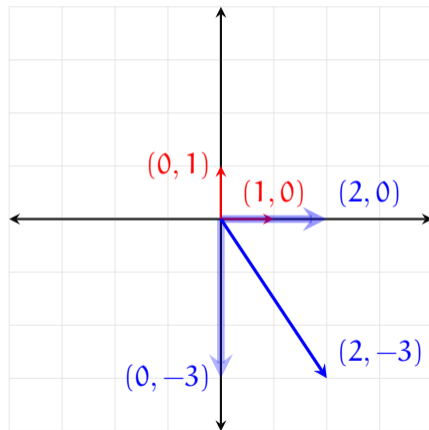- n-dimensional standard unit vectors form a *basis* for $\mathbb{R}^n$

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Dot (inner) product

- Dot product is an operation between two vectors with same dimensions

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

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- Calculate the dot products for the following vectors

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Dot (inner) product

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- Note that dot product is larger when the vectors are 'similar'

Properties of dot product

- *Commutativity* $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- *Distributivity with vector addition* $\mathbf{u} \cdot (\mathbf{v} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u}$
- *Associativity with scalar multiplication* $(a\mathbf{u}) \cdot (b\mathbf{v}) = ab(\mathbf{u} \cdot \mathbf{v})$.
- Note that dot product is not associative, since the result of the dot product is not a vector, but a scalar
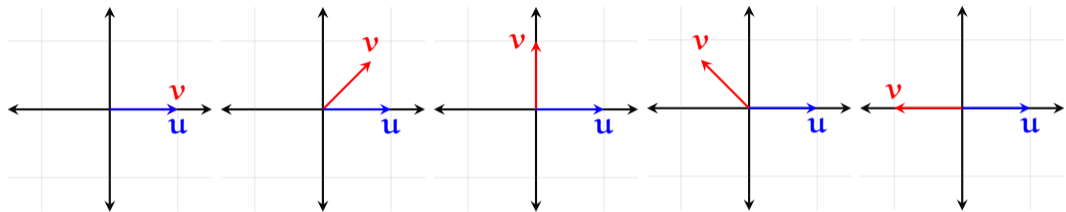
Geometric interpretation of the dot product

- The dot product of two vectors gives
  the (orthogonal) projection of one of
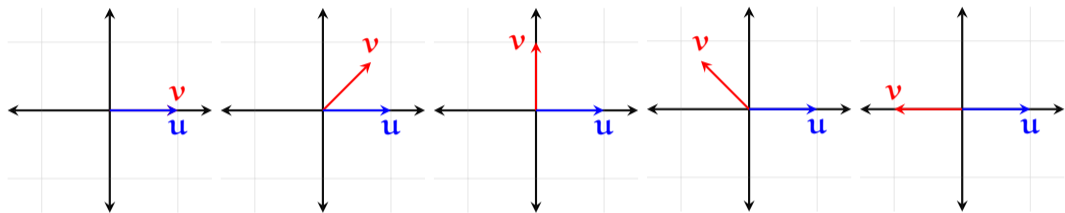 the vectors to the line defined by the
  other
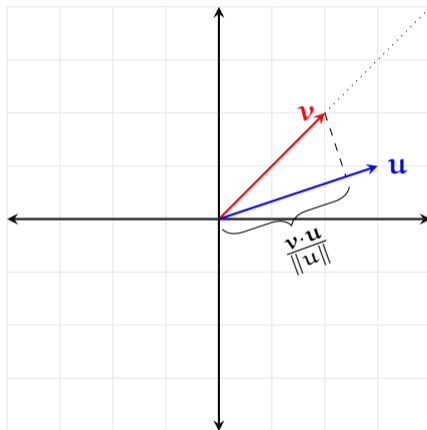
Dot product with unit vectors

Dot product with unit vectors

$$u \cdot v = 1$$

Dot product with unit vectors

$\mathbf{u} \cdot \mathbf{v} = 1$ $\mathbf{u} \cdot \mathbf{v} = 0.7$

Dot product with unit vectors

$$\mathbf{u} \cdot \mathbf{v} = 1 \qquad \mathbf{u} \cdot \mathbf{v} = 0.7 \qquad \mathbf{u} \cdot \mathbf{v} = 0$$

Dot product with unit vectors

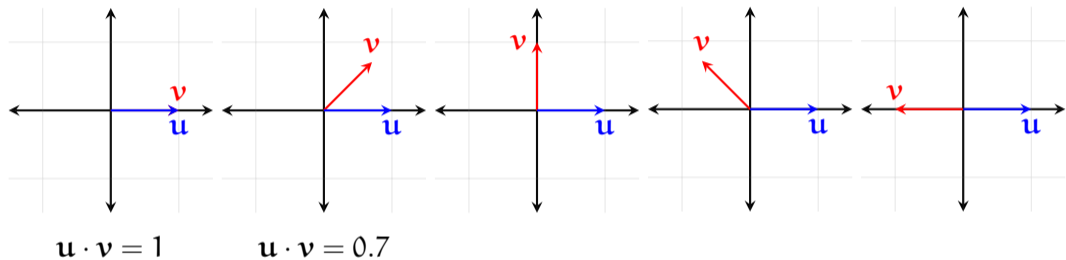$\mathbf{u} \cdot \mathbf{v} = 1$ $\mathbf{u} \cdot \mathbf{v} = 0.7$ $\mathbf{u} \cdot \mathbf{v} = 0$ $\mathbf{u} \cdot \mathbf{v} = -0.7$
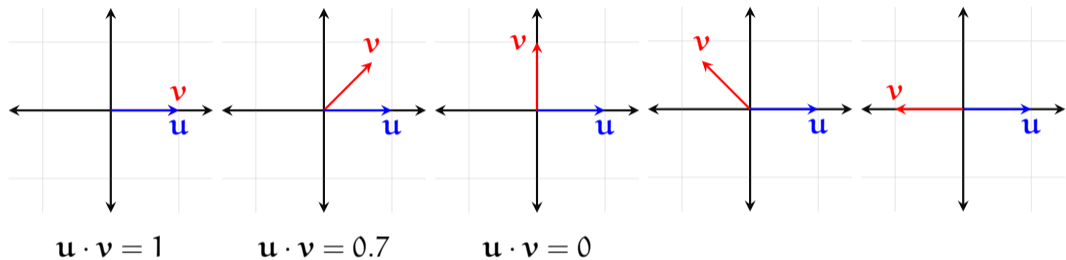
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Dot product with unit vectors

$\mathbf{u} \cdot \mathbf{v} = 1$ $\mathbf{u} \cdot \mathbf{v} = 0.7$ $\mathbf{u} \cdot \mathbf{v} = 0$ $\mathbf{u} \cdot \mathbf{v} = -0.7$ $\mathbf{u} \cdot \mathbf{v} = -1$
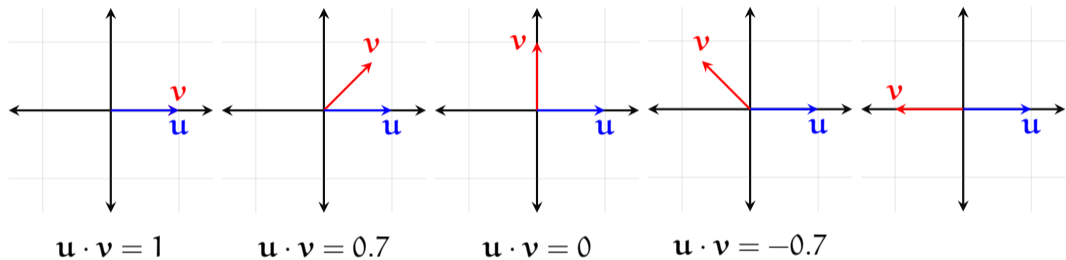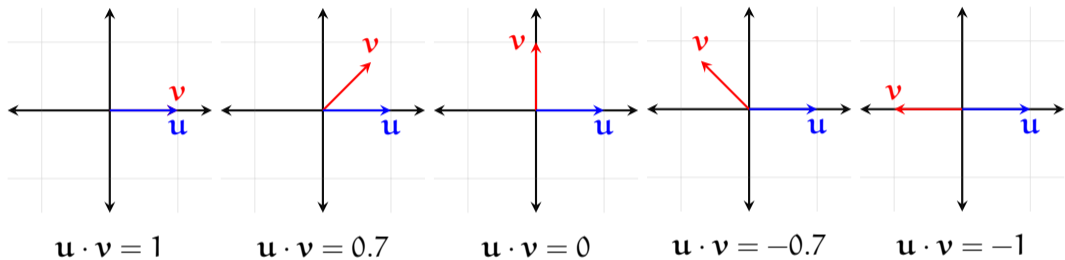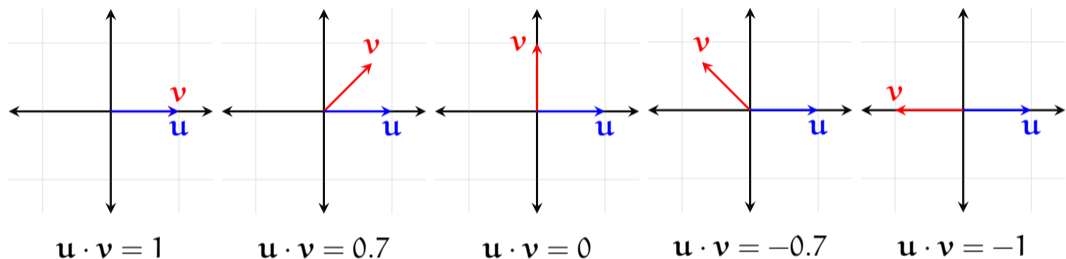
- The dot product is larger if the vectors point to the similar directions

Vector norms

- The *norm* of a vector is an indication of its size (magnitude)
- The norm of a vector is the distance from its tail to its tip
- Norms are related to distance measures
- Vector norms are particularly important for understanding some machine learning techniques

L2 norm

- Euclidean norm, or L2 (or L_2) norm is the most commonly used norm

- For $v = (v_1, v_2, \ldots v_n)$,

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \ldots v_n}$$
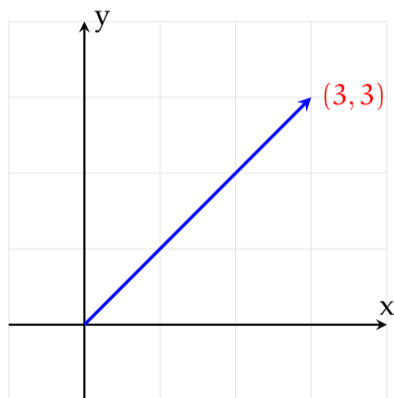$$= \sqrt{v \cdot v}$$

- For example,

$$\|(3,3)\|_2 = \sqrt{3^2 + 3^2} = \sqrt{18}$$

- L2 norm is the default, we often skip the subscript $\|v\|$

Euclidean distance

- Euclidean distance between two
  vectors is the L2 norm of their
  difference

$$D(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-6)^2 + (-1)^2}$$

- Euclidean distance is a metric
  - symmetric $\|\mathbf{v} - \mathbf{u}\| = \|\mathbf{u} - \mathbf{v}\|$
  - non-negative
  - and obeys the triangle inequality
    $D(\mathbf{u}, \mathbf{v}) \leqslant D(\mathbf{u}, \mathbf{w}) + D(\mathbf{w}, \mathbf{v})$
    for any $\mathbf{w}$
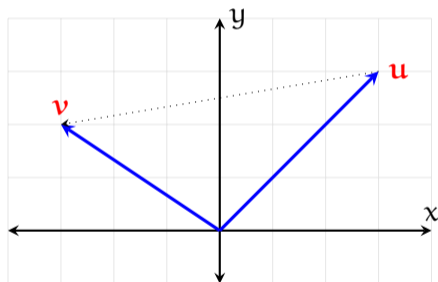
Cauchy–Schwarz inequality

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leqslant \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

- In words: the product of the norms of two vectors is greater than or equal to absolute value of their dot product

Cosine similarity

- The cosine of the angle between two vectors

$$\cos \theta = \frac{v \cdot u}{\|v\| \cdot \|u\|}$$

  is called *cosine similarity*

- Unlike dot product, the cosine similarity is not sensitive to the magnitudes of the vectors

- The cosine similarity is bounded in range $[-1, +1]$
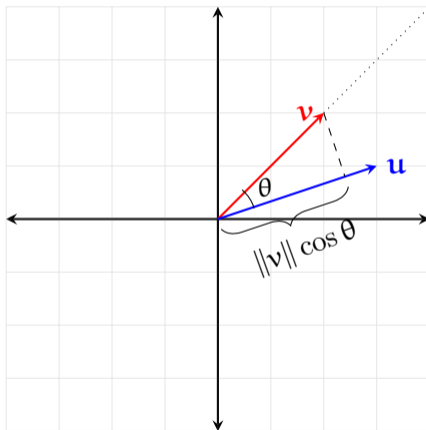
L1 norm

- Another norm we will often
  encounter is the L1 norm

$$\|v\|_1 = |v_1| + |v_2|$$

$$\|(3,3)\|_1 = |3| + |3| = 6$$

- L1 norm is related to Manhattan
  distance
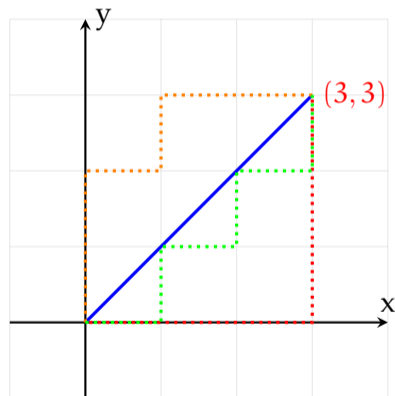
L$_P$ norm

In general, L$_P$ norm, is defined as

$$\|\boldsymbol{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}}$$

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We will only work with than L1 and L2 norms, but you may also see L_0 and L_∞ norms in related literature

Matrices

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,m} \end{bmatrix}$$

- We can think of matrices as collection of row or column vectors
- A matrix with n rows and m columns is in $\mathbb{R}^{n \times m}$
- Most operations in linear algebra also generalize to more than 2-D objects
- A *tensor* can be thought of a generalization of vectors and matrices to multiple dimensions

Matrices

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Multiplying a matrix with a scalar

Similar to vectors, each element is multiplied by the scalar.

$$2 \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 \times 2 & 2 \times 1 \\ 2 \times 1 & 2 \times 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}$$

Matrix addition and subtraction

Each element is added to (or subtracted from) the corresponding element

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

Note:

- Matrix addition and subtraction are defined on matrices of the same dimensions

Transpose of a matrix

Transpose of a $n \times m$ matrix is an $m \times n$ matrix whose rows are the columns of the original matrix.

Transpose of a matrix $\mathbf{A}$ is denoted with $\mathbf{A}^\top$.

$$\text{If } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \mathbf{A}^\top = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}.$$

Some special matrices
Identity matrix

- A square matrix in which all the elements of the principal diagonal are one and all other elements are zero is called *identity matrix* (**I**)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Multiplying a vector and matrix with the identity matrix has no affect

Some special matrices
Diagonal matrices

- Diagonal matrices are similar to $\mathbf{I}$. All non-diagonal entries are 0s, but non-zero entries can only be in the main diagonal
- Example:

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Some special matrices
Upper/lower triangular matrices

- Triangular matrices are common in many applications
- An upper triangular matrix have all 0s below main diagonal. Example:

$$\begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- An lower triangular matrix have all 0s above main diagonal. Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 8 & 7 & 1 \end{bmatrix}$$

Symmetric matrices

- Symmetric matrices arise in many applications, including in ML/NLP (e.g., similarity or distance matrices)
- A symmetric matrix $\mathbf{A}$ satisfies $a_{ij} = a_{ji}$, or $\mathbf{A} = \mathbf{A}^{\top}$
- Example:

$$\begin{bmatrix} 1 & 4 & -2 \\ 4 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$

- Symmetric matrices have some interesting properties (that we will return later)

Matrix–vector multiplication

- An $n \times m$ matrix can be multiplied with a m-vector to yield a n-vector

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- Example

$$
\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \times 0 + 1 \times 1 + 0 \times 1 \\ 1 \times 0 + 0 \times 1 + 1 \times 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

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- One view of this operation: each entry in the resulting vector is a dot product (of rows of the matrix and the vector)

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- One view of this operation: each entry in the resulting vector is a dot product (of rows of the matrix and the vector)
- Another: the result is a linear combination of the columns of the matrix (with the entries in the vector as coefficients)

$$0 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Matrix multiplication as linear transformation

- Multiplying a vector with a matrix transforms the vector
- The result is another vector (possibly in a different vector space)
- Many operations on vectors can be expressed with multiplying with a matrix (linear transformations)

Transformation examples
identity

- Identity transformation maps a vector to itself
- For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transformation examples

permutation

- Another simple transformation is to permute (re-arrange) the elements (rows) of the vector
- For example:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

Transformation examples
stretch along the x axis

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Transformation examples
rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Transformation examples
rotation

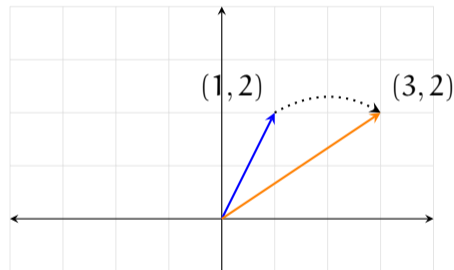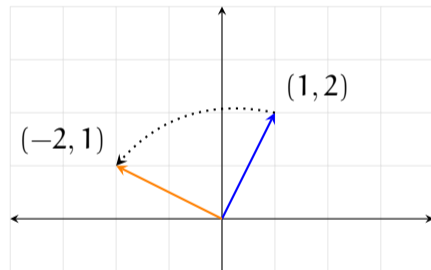$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Transformations by rectangular matrices

- Multiplying a vector with (compatible) rectangular matrix results in a vector with different dimensionality

- Example $\mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Example $\mathbb{R}^3 \to \mathbb{R}^4$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Dot product as matrix multiplication

In machine learning (and many other disciplines, we treat an n-vector as $n \times 1$ matrix.

Then, the *dot product* of two vectors is

$$\mathbf{u}^\mathsf{T}\mathbf{v}$$

For example, $\mathbf{u} = (2, 2)$ and $\mathbf{v} = (2, -2)$,

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Dot product as matrix multiplication

In machine learning (and many other disciplines, we treat an n-vector as $n \times 1$ matrix.

Then, the *dot product* of two vectors is

$$u^\mathsf{T} v$$

For example, $u = (2, 2)$ and $v = (2, -2)$,

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \times 2 + 2 \times -2 = 4 - 4 = 0$$

- This is a 1×1 matrix, but matrices and vectors with single entries are often treated as scalars

> Question: What is the transformation performed by dot product?

Outer product

The *outer product* of two column vectors is defined as

$$\mathbf{v}\mathbf{u}^\mathsf{T}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} =$$

Outer product

The *outer product* of two column vectors is defined as

$$vu^\mathsf{T}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Note:

- The result is a matrix
- The vectors do not have to be the same length

Matrix multiplication

- if A is a $n \times k$ matrix, and B is a $k \times m$ matrix, their product C is a $n \times m$ matrix
- Elements of C, $c_{i,j}$, are defined as

$$c_{ij} = \sum_{\ell=0}^{k} a_{i\ell} b_{\ell j}$$

- Note: $c_{i,j}$ is the dot product of the ith row of A and the jth column of B

Matrix multiplication

(demonstration)

$$
\begin{pmatrix}
a_{11} & a_{12} & \dots & a_{1k} \\
a_{21} & a_{22} & \dots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nk}
\end{pmatrix}
\times
\begin{pmatrix}
b_{11} & b_{12} & \dots & b_{1m} \\
b_{21} & b_{22} & \dots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k1} & b_{k2} & \dots & b_{km}
\end{pmatrix}
$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \dots a_{1k}b_{k1}$$

$$
=
\begin{pmatrix}
c_{11} & c_{12} & \dots & c_{1m} \\
c_{21} & c_{22} & \dots & c_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \dots & c_{nm}
\end{pmatrix}
$$

Matrix multiplication
(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \dots a_{1k}b_{k2}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Matrix multiplication

(demonstration)

$$
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1k} \\
a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nk}
\end{pmatrix}
\times
\begin{pmatrix}
b_{11} & b_{12} & \ldots & b_{1m} \\
b_{21} & b_{22} & \ldots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k1} & b_{k2} & \ldots & b_{km}
\end{pmatrix}
$$

$$c_{1m} = a_{11}b_{1m} + a_{12}b_{2m} + \ldots a_{1k}b_{km}$$

$$
=
\begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1m} \\
c_{21} & c_{22} & \ldots & c_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \ldots & c_{nm}
\end{pmatrix}
$$

Matrix multiplication

(demonstration)

$$
\left(
\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1k} \\
a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nk}
\end{array}
\right)
\times
\left(
\begin{array}{cccc}
b_{11} & b_{12} & \ldots & b_{1m} \\
b_{21} & b_{22} & \ldots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k1} & b_{k2} & \ldots & b_{km}
\end{array}
\right)
$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \ldots a_{2k}b_{k1}$$

$$
=
\left(
\begin{array}{cccc}
c_{11} & c_{12} & \ldots & c_{1m} \\
c_{21} & c_{22} & \ldots & c_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \ldots & c_{nm}
\end{array}
\right)
$$

Matrix multiplication
(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \ldots & b_{km} \end{pmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + \ldots a_{2k}b_{k2}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1m} \\ c_{21} & c_{22} & \ldots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nm} \end{pmatrix}$$

Matrix multiplication
(demonstration)

$$
\begin{pmatrix}
a_{11} & a_{12} & \dots & a_{1k} \\
a_{21} & a_{22} & \dots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nk}
\end{pmatrix}
\times
\begin{pmatrix}
b_{11} & b_{12} & \dots & b_{1m} \\
b_{21} & b_{22} & \dots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k1} & b_{k2} & \dots & b_{km}
\end{pmatrix}
$$

$$
c_{2m} = a_{21}b_{1m} + a_{22}b_{2m} + \dots a_{2k}b_{km}
$$

$$
=
\begin{pmatrix}
c_{11} & c_{12} & \dots & c_{1m} \\
c_{21} & c_{22} & \dots & c_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \dots & c_{nm}
\end{pmatrix}
$$

Matrix multiplication
(demonstration)

$$
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1k} \\
a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nk}
\end{pmatrix}
\times
\begin{pmatrix}
b_{11} & b_{12} & \ldots & b_{1m} \\
b_{21} & b_{22} & \ldots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k1} & b_{k2} & \ldots & b_{km}
\end{pmatrix}
$$

$$
c_{n1} = a_{n1}b_{11} + a_{n2}b_{22} + \ldots a_{nk}b_{k1}
$$

$$
=
\begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1m} \\
c_{21} & c_{22} & \ldots & c_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \ldots & c_{nm}
\end{pmatrix}
$$

Matrix multiplication

(demonstration)

$$
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1k} \\
a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nk}
\end{pmatrix}
\times
\begin{pmatrix}
b_{11} & b_{12} & \ldots & b_{1m} \\
b_{21} & b_{22} & \ldots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k1} & b_{k2} & \ldots & b_{km}
\end{pmatrix}
$$

$$c_{n2} = a_{n1}b_{12} + a_{n2}b_{22} + \ldots a_{nk}b_{k2}$$

$$
=
\begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1m} \\
c_{21} & c_{22} & \ldots & c_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \ldots & c_{nm}
\end{pmatrix}
$$

Matrix multiplication

(demonstration)

$$\left(\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} \end{array} \right) \times \left(\begin{array}{cccc} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \ldots & b_{km} \end{array} \right)$$

$$c_{nm} = a_{n1}b_{1m} + a_{n2}b_{2m} + \ldots a_{nk}b_{km}$$

$$= \left(\begin{array}{cccc} c_{11} & c_{12} & \ldots & c_{1m} \\ c_{21} & c_{22} & \ldots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nm} \end{array} \right)$$

Matrix multiplication
(demonstration)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{pmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots a_{ik}b_{kj}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{pmatrix}$$

Properties of matrix multiplication

- Associativity

$$(\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C})$$

- Distributivity

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C}$$

- Multiplication by Identity

$$\mathbf{I}\mathbf{A} = \mathbf{A}\mathbf{I} = \mathbf{A}$$

- Matrix multiplication is not commutative $\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$ (in general)
- Matrix multiplication and transpose

$$(\mathbf{A}\mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$$

Question

We have three matrices:

- **A**: a 10×2 matrix
- **B**: a 2×5 matrix
- **C**: a 5×10 matrix
- What is the dimensionality of $\mathbf{ABC}$

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Question

We have three matrices:

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Question

We have three matrices:

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- **C**: a 5×10 matrix

- What is the dimensionality of **ABC**
- Does it matter if we perform the multiplication as
  - (**AB**)**C**, or
  - **A**(**BC**)

Alternative ways to think about matrix multiplication

If we have $AB = C$,

- Column vectors of C, $c_j = Ab_j$
- Row vectors of C, $c_i^\top = a_i^\top B$
- C is also the sum of outer product of columns of A and rows of B

$$C = \sum a_i b_i^\top$$

Matrix multiplication example

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} =$$

Matrix multiplication example

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$$

Matrix-vector representation of a set of linear equations

The set of linear equations

$$
\begin{aligned}
2x_1 &+ x_2 &= 6 \\
x_1 &+ 4x_2 &= 17
\end{aligned}
$$

can be written as:

$$
\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 17 \end{bmatrix}}_{b}
$$

One can solve the above equation using *Gaussian elimination* (we will not cover it today).

Summary & next week

- Vectors, matrices
- Dot product
- Next: solving systems of linear equations

Further reading

- A classic reference book in the field is Strang (2009)
- Shifrin and Adams (2011) and Farin and Hansford (2014) are textbooks with a more practical/graphical orientation.
- Cherney, Denton, and Waldron (2013) and Beezer (2014) are two textbooks that are freely available.
- Form more alternatives, see
  http://www.openculture.com/free-math-textbooks
- You may also find the MIT video lectures on introductory linear algebra at
  https://www.youtube.com/playlist?list=PL49CF3715CB9EF31D

Beezer, Robert A. (2014). *A First Course in Linear Algebra*. version 3.40. Congruent Press. ISBN: 9780984417551. URL: http://linear.ups.edu/.

Cherney, David, Tom Denton, and Andrew Waldron (2013). *Linear algebra*. math.ucdavis.edu. URL: https://www.math.ucdavis.edu/~linear/.

Farin, Gerald E. and Dianne Hansford (2014). *Practical linear algebra: a geometry toolbox*. Third edition. CRC Press. ISBN: 978-1-4665-7958-3.

Shifrin, Theodore and Malcolm R Adams (2011). *Linear Algebra. A Geometric Approach*. 2nd. W. H. Freeman. ISBN: 978-1-4292-1521-3.

Further reading (cont.)

Strang, Gilbert (2009). *Introduction to Linear Algebra, Fourth Edition*. 4th ed. Wellesley Cambridge Press. ISBN: 9780980232714.